PH 5840 - Lecture 1

Note Title 8/3/2015

Curtain-raiser: Qubits, Bloch sphere, Q Parallelism

① "Information is physical" - Rolf Landauer

- Classical bit $x \in \{0,1\} \longrightarrow$ Quantum bit
  $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$
  $|\alpha|^2 + |\beta|^2 = 1.$

(Ben Schumacher, 1995)

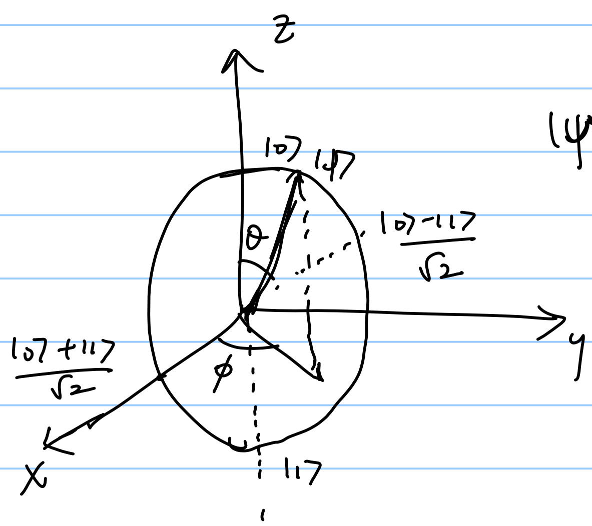
- Qubit : $|\psi\rangle \in \mathbb{C}^2$ (Complex v.space of dim. 2)
  (unit-vector)
  $\Rightarrow$ Spanned by a pair of orthonormal states

"Computational" Basis : $\{|0\rangle, |1\rangle\}$

Eg. spin-½ system : $|+\frac{1}{2}\rangle$, $|-\frac{1}{2}\rangle$
photon-polarization : $|H\rangle$, $|V\rangle$

* Matrix notation:
$|0\rangle = \begin{pmatrix}1\\0\end{pmatrix}$, $|1\rangle = \begin{pmatrix}0\\1\end{pmatrix}$ $\rightarrow$ $|\psi\rangle$ is a 2-d column vector with complex entries. $\begin{pmatrix}\alpha\\\beta\end{pmatrix}$

- Geometric representation of a qubit : Bloch/Poincare sphere



$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi/2}\sin\frac{\theta}{2}|1\rangle$$

"Hilbert space is a big place"
– Carl Caves

- Alternate basis: $|+\rangle = \dfrac{|0\rangle + |1\rangle}{\sqrt{2}}$

$$|-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

(2) Logical operations: Unitary operators (Quantum gates)

(a) NOT gate: $\left.\begin{array}{l} 0 \rightarrow 1 \\ 1 \rightarrow 0 \end{array}\right\}$ bit-flip

Quantum NOT gate: $|0\rangle \rightarrow |1\rangle$
$|1\rangle \rightarrow |0\rangle$

Matrix notation: $|0\rangle \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $|1\rangle \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

NOT: $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \equiv \sigma_x$ Pauli X-operator

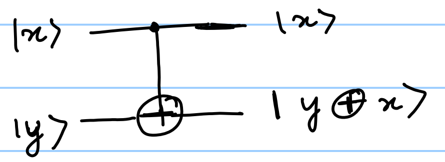
(b) Why $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ "Phase-flip" operator

(c) Basis change: $\{|0\rangle, |1\rangle\} \rightleftarrows \{|+\rangle, |-\rangle\}$

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

(d) Multiqubit gates: CNOT

$|00\rangle \rightarrow |00\rangle$, $|01\rangle \rightarrow |01\rangle$, $|10\rangle \rightarrow |11\rangle$
$|11\rangle \rightarrow |10\rangle$

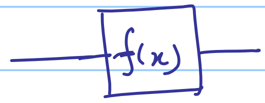


$$|x, y\rangle \xrightarrow[\text{CNOT}]{} |x, y \oplus x\rangle$$

③ Power of superposition: Deutsch algorithm

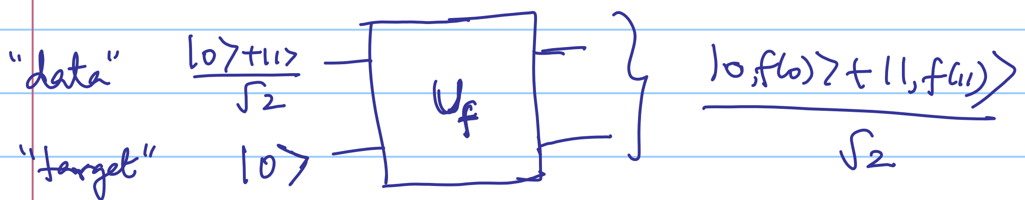
- Given binary function $f(x)$: $f(0) = f(1)$ constant
  $f(0) \neq f(1)$ balanced.
- To check if $f(x)$ is constant or balanced:
  - compute both values in a classical computer
  - compute a superposition ("intermediate" value) in a quantum computer!

Proc. R. Soc. Lon. A 400:97, 1985

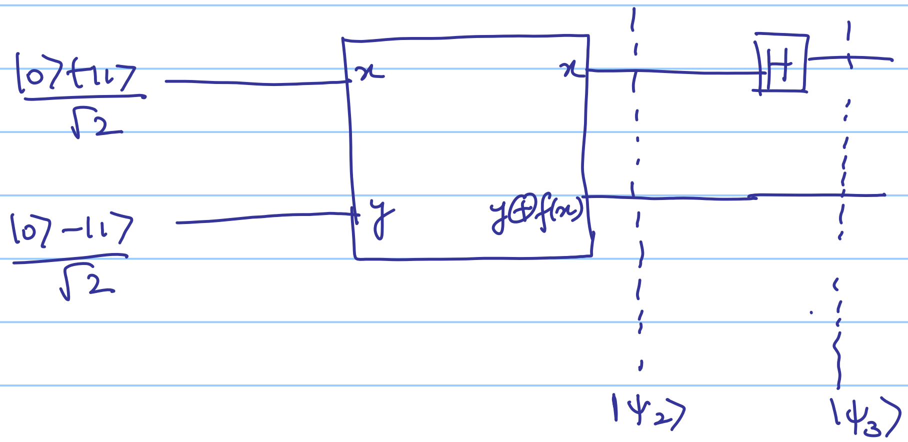


$U_f : |x, y\rangle \mapsto |x, y \oplus f(x)\rangle$

$\downarrow$



* Solving Deutsch's problem:-



(i) $|\psi_1\rangle = \left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right)\left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right) = \frac{|00\rangle + |10\rangle - |01\rangle - |11\rangle}{2}$

(ii) $$|\psi_2\rangle = \frac{|0, f(0)\rangle + |1, f(1)\rangle - |0, 1\oplus f(0)\rangle - |1, 1\oplus f(1)\rangle}{2}$$

ⓐ If $f(0)=f(1)$, $$|\psi_2\rangle = \frac{|0\rangle}{\sqrt{2}}\left(\frac{|f(0)\rangle - |\overline{f(0)}\rangle}{\sqrt{2}}\right) + \frac{|1\rangle}{\sqrt{2}}\left(\frac{|f(0)\rangle - |\overline{f(0)}\rangle}{\sqrt{2}}\right)$$

$$= \pm\left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right)\left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right)$$

ⓑ If $f(0)\neq f(1)$, $$|\psi_2\rangle = \frac{|0\rangle}{\sqrt{2}}\left(\frac{|f(0)\rangle - |\overline{f(0)}\rangle}{\sqrt{2}}\right) + \frac{|1\rangle}{\sqrt{2}}\left(\frac{|\overline{f(0)}\rangle - |f(0)\rangle}{\sqrt{2}}\right)$$

$$= \pm\left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right)\left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right)$$

<u>Phase matters!</u>

(iii) Extract the phase information using H gate

$$|\psi_3\rangle = \begin{cases} \pm|0\rangle\left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right) & \text{if } f(0)=f(1) \\ \pm|1\rangle\left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right) & \text{if } f(0)\neq f(1) \end{cases}$$

$$= \pm|f(0)\oplus f(1)\rangle\left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right)$$

∴ Evaluated a global property of $f(x)$ using simply one evaluation!

<u>Twice</u> as fast as a classical device!

* Scaling it up: (Query complexity)

ⓐ D-J: Say $f(x)$ is a n-bit binary function:

Proc. Roy. Soc. A, 439: 553 1992

$x = x_1 x_2 x_3, \ldots x_n \quad f(x) \in \{0,1\}$

To check if $f(x)$ is constant or balanced:

- classically, need to perform: $\frac{2^n}{2}+1$ evaluations "Queries"

Quantum device: just 1 evaluation! → "Query"

"Exponential speedup".

→ Practical problems?

ⓑ Search problem: Searching thro' N entries (unstructured)

classically - N

Quantumly - $\sqrt{N}$ (Grover's algorithm)

ⓒ Prime-factorization:

Factoring a N-digit number:

classical - $N \log N$; Quantum: $(\log N)^2$.

(Shor's algorithm)

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* References: ⓐ Nielsen & Chuang, chapter-1

ⓑ Feynman (1982): Simulating quantum mechanical systems on classical computers

"Simulating Physics with computers", IJTP, 21 (6,7) 1982

* Church-Turing Thesis:

" Any algorithmic process can be simulated efficiently using a (probabilistic) Turing machine."